FEM-Source Integral Boundary Conditions for Computation of the Open-Boundary Electrostatic Fields

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Abstract — This paper extends a method to solve the unbounded electrostatic fields. The method is based on the evaluation of the potential on some fictitious boundaries enclosing each part of the conductors and dielectrics, according to the charge lying on their surface, or the dipole lying in the dielectrics volume. The major advantage of this method lies in its high efficiency of computing the electrostatic field in which sources and materials spread out at a relatively long distance.

I. Introduction

In the context of the finite element method (FEM) a variety of techniques have been proposed to compute the unbounded electrostatic fields. Truncation is the simplest among them. However, an onerous discretization of a wide domain is generally required if accurate results are desired. Hybrid methods combine FEM with other methods to solve the unbounded problem. The unbounded problem is divided into two regions by a fictitious boundary. The inner region is solved by FEM, and the outer region is solved by other methods. Finite element method-boundary element method (FEM-BEM) [1]-[2] is one of the hybrid methods, which is popular in the electrostatic field. The boundary integral equations in BEM describe the outer region as equivalent charge and dipole, lying on the boundary, and calculate the Dirichlet conditions by integral of them. Because the source points and field points are all on the same boundary, when the distance is close to zero, the calculation error could be great. FEM- Dirichlet boundary condition iteration (DBCI) [3]-[4] is extended to integral on another boundary, which is surrounded by the fictitious boundary wholly. In this paper we advanced a new hybrid method, FEM-source integral boundary conditions [5]. In the method, the materials in electrostatic field are replaced by equivalent charge or dipole. We calculate the Dirichlet conditions by integral of all of the real sources and equivalent sources.

II. The FEM-Source Integral Boundary Conditions

The unbounded electrostatic fields are determined by charge and materials in the vacuum. After the materials are replaced by equivalent sources, the electrostatic field can be determined by real charge and equivalent sources in the vacuum.

The FEM-source integral boundary conditions method is illustrated by the simple 3-D system shown in Fig. 1. It's an electrostatic system. Fictitious boundaries enclosed several parts separately. In the bounded domain D_i, the Laplace's equation is

$$-\varepsilon\nabla^2\varphi = 0 \tag{1}$$

where ε is the electric permittivity and φ is the scalar electrical potential in the bounded domain. In FEM, (1) can be discretized as

$$\mathbf{K}\boldsymbol{\varphi} = \mathbf{0}\,. \tag{2}$$

where $\mathbf{K}$ is a sparse matrix.

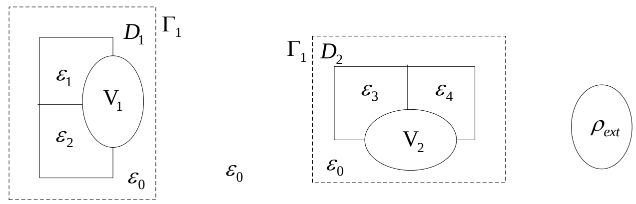

Fig. 1. An electrostatic system of voltaged conductors and non-homogeneous dielectric objects is enclosed by fictitious boundaries.

To calculate (2) we need boundary conditions on Γ_1. The boundary conditions are expressed by $\varphi|_{\Gamma_1}$. To calculate the $\varphi|_{\Gamma_1}$, the materials should be equaled to equivalent sources. Voltaged conductors could be replaced by free charge, lying on its surface. Dielectrics could be replaced either by bound charge, lying on its surface, or by dipole, lying in its volume. The $\varphi|_{\Gamma_1}$ will be determined by integral of all of the sources. The sources include real charges, equivalent charges of the conductors, and equivalent sources of the dielectrics. The integral equations will be discussed in next section.

There are two ways to calculate (2) with $\varphi|_{\Gamma_1}$.

One is the couple of FEM and integral equations. It can be described as

$$\begin{bmatrix} \mathbf{K_1} & \mathbf{K_2} \\ \mathbf{C_1} & \mathbf{C_2} \end{bmatrix}\begin{bmatrix} \boldsymbol{\varphi} \\ \boldsymbol{\varphi}|_{\boldsymbol{\Gamma_1}} \end{bmatrix} = [\mathbf{0}] \tag{3}$$

where $\mathbf{K}_1\boldsymbol{\varphi} + \mathbf{K}_2\boldsymbol{\varphi}|_{\boldsymbol{\Gamma_1}} = \mathbf{0}$ is a part of (2) without the equations on boundary nodes; $\mathbf{C}_1\boldsymbol{\varphi} + \mathbf{C}_2\boldsymbol{\varphi}|_{\boldsymbol{\Gamma_1}} = \mathbf{0}$ is the boundary integral equation.

The other is iteration. At first, we let

$$\varphi|_{\Gamma_1}{}^{(1)} = \boldsymbol{\varphi_0}^{(0)} \tag{4}$$

where $\boldsymbol{\varphi_0}^{(0)}$ is arbitrary constant column. It's the initial value of $\boldsymbol{\varphi}|_{\boldsymbol{\Gamma_1}}$. Step 2 solves the bounded domains by FEM. Step 3 calculates equivalent sources of the materials. Step 4 calculates new Dirichlet conditions by boundary integral equation. Step 5 returns to the step 2 with new boundary conditions. The problem is iterated until convergence. The procedure can be described as

$$\begin{bmatrix} \mathbf{K_1} & \mathbf{K_2} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \boldsymbol{\varphi}^{(i)} \\ \boldsymbol{\varphi}\big|_{\Gamma_1}{}^{(i)} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{\varphi}_{\mathbf{0}}{}^{(i-1)} \end{bmatrix} \quad (i=1,2,\ldots,n) \tag{5}$$

where i indicates the FEM calculation time and $\varphi\big|_{\Gamma_1}{}^{(i)}$ is determined by equivalent sources, according $\varphi^{(i-1)}$.

The iteration is more efficiency than coupling when the interested problem is complex.

III. Boundary Integral Equations

In electrostatic fields conductor can be replaced by equivalent charge, lying on its surface. The conductor's contribution to Dirichlet conditions is

$$\varphi\big|_{\Gamma_1} = \frac{1}{4\pi\varepsilon_0} \oiint \left(-\varepsilon_0 \frac{\partial\varphi}{\partial n}\right) \frac{1}{R_C} dS_C{}' \tag{6}$$

where $S_C{}'$ is the surface of conductor, R_C is the distance from a source point on $S_C{}'$ to the field point on Γ_1, and φ is the potential in vacuum or dielectric next to the surface of conductor.

In electrostatic fields the dielectric could be considered as bound charge with surface charge density σ_P. For homogeneous, linearity and isotropy dielectric,

$$\sigma_P = (\varepsilon_0 - \varepsilon) \frac{\partial\varphi}{\partial n} \tag{7}$$

where φ is the potential in the dielectric next to its surface and $\boldsymbol{e}_n$ is a unit vector normal to the dielectric surface. Therefore, the dielectric's contribution to Dirichlet conditions is

$$\varphi\big|_{\Gamma_1} = \frac{1}{4\pi\varepsilon_0} \oiint (\varepsilon_0 - \varepsilon) \frac{\partial\varphi}{\partial n} \frac{1}{R_D} dS_D{}' \tag{8}$$

where $S_D{}'$ is the surface of dielectric, R_D is the distance from a source point on $S_D{}'$ to the field point on Γ_1, and φ is the potential in dielectric next to its surface.

Another method to calculate the dielectric's contribution to Dirichlet conditions is

$$\varphi\big|_{\Gamma_1} = \iiint \frac{\mathbf{P}\cdot\mathbf{e}_{R_D}}{4\pi\varepsilon_0 R_D{}^2} dV_D{}' \tag{9}$$

where polarization $\mathbf{P}$ is regarded as volume density of dipole moment, $V_D{}'$ is the volume of dielectric and $\boldsymbol{e}_{R_D}$ is a unit vector of R_D, which is the distance from the source point in $V_D{}'$ to the field point on Γ_1.

The boundary integral equation is

$$\begin{aligned} \varphi\big|_{\Gamma_1} &= \oiint \left(-\varepsilon_0 \frac{\partial\varphi}{\partial n}\right) \frac{1}{4\pi\varepsilon_0 R_C} dS_C{}' \\ &+ \oiint (\varepsilon_0 - \varepsilon) \frac{\partial\varphi}{\partial n} \frac{1}{4\pi\varepsilon_0 R_D} dS_D{}' + \iiint \frac{\rho_{ext}}{4\pi\varepsilon_0 R_{ext}} dV' \end{aligned} \tag{10}$$

or

$$\begin{aligned} \varphi\big|_{\Gamma_1} &= \frac{1}{4\pi\varepsilon_0} \oiint \left(-\varepsilon_0 \frac{\partial\varphi}{\partial n}\right) \frac{1}{R_C} dS_C{}' + \iiint \frac{\mathbf{P}\cdot\mathbf{e}_{R_D}}{4\pi\varepsilon_0 R_D{}^2} dV_D{}' \\ &+ \frac{1}{4\pi\varepsilon_0} \iiint \frac{\rho_{ext}}{R_{ext}} dV' \end{aligned} . \tag{11}$$

IV. Validation Examples

A. Voltaged Conductor Sphere Enclosed by Dielectric Shell

The first example concerns the computation of the potential φ near a conductor sphere. It is voltaged at V_0. There's a dielectric shell enclosing the conductor, and they are concentric. They are all embedded in vacuum. We solved the problem by 3-D model, and then compared the result with analytical solution.

B. Volume Charge and Dielectric Block

The second example concerns the computation of the potential φ in a dielectric block. The dielectric's permittivity is ε. There are two volume charges. One is above the dielectric block with volume charge density ρ_1 and the other is under the dielectric block with volume charge density ρ_2, as shown in Fig. 2.

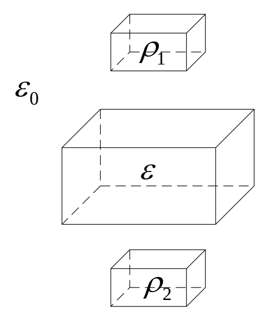

Fig. 2. A dielectric block with the presence of volume charges.

V. References

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